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Two-mode squeezing in a coherently driven degenerate parametric down conversion

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Abstract. Squeezing properties of a two-mode radiation produced by a process of driven degenerate parametric down conversion, when the cavity is coupled to two independent squeezed vacuum reservoirs employing the linearization procedure, are analyzed. The two-mode cavity and output radiations exhibit considerable squeezing even when the oscillator is coupled to a vacuum reservoir. One of the effects of coupling the cavity to the squeezed vacuum reservoirs is to increase the degree of squeezing exponentially. For the output radiation the correlation of the quadrature operators evaluated at different times also contributes to the squeezing, which is the reason for quenching of the overall noise in one of the quadrature components of the fundamental mode.

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1 Introduction

Optical degenerate parametric down conversion, in which a pump photon of frequency 2ω is down converted by a nonlinear crystal into a pair of signal photons each of frequency ω, is one of the most interesting and well studied phenomena in the nonlinear quantum optics [1–15]. As it has been shown earlier, due to the inherent two-photon nature of the interaction, the parametric oscillator can be taken as a conventional source of a squeezed light [1]. The maximum achievable squeezing for the single-mode cavity radiation coupled to an ordinary vacuum is found to be 50% by many authors following various approaches [1–4]. Most recently, employing the state of the art squeezed state generation based on periodically poled nonlinear crystal, Takeno et al. [5] obtained a single-mode squeezing larger than 85% for output radiation in a parametric down conversion process. The limitation in the degree of squeezing is mainly attributed to the transmission through the mirror and amplification of the noise in the cavity, since the vacuum field has no definite phase. However, if the ordinary vacuum reservoir is replaced by a squeezed vacuum, the noise fluctuations entering the cavity would be biased and hence the squeezing of the cavity radiation could be enhanced provided that the reservoir is squeezed in the right quadrature. In this respect, the squeezing of the cavity radiation of a degenerate parametric oscillator coupled to a single-mode squeezed vacuum reservoir is found to increase exponentially with the squeezed input [2].

It is not difficult to envision that when a nonlinear crystal is shined with an external radiation of frequency 2ω, only some part of this radiation would be down converted into a pair of photons with frequency ω. That is, the cavity contains a two-mode radiation that comprises the down converted and unchanged radiation which I designated as the fundamental and second-harmonic modes [6,7], respectively. It is a common knowledge that analysis of the squeezing properties and photon statistics of the fundamental mode of the degenerate parametric oscillator coupled to a squeezed vacuum reservoir has been exhaustively made over the years by many authors [1–3] upon treating the second-harmonic mode classically. Although this consideration appears to be mathematically friendly, it denies the very existence of the two-mode light in the cavity. Contrary to this well established claim, in recent years different authors showed that the second-harmonic mode also exhibits squeezing following various approaches [6,8,9]. Due to the coherence in the driving mechanism, it is reasonable to expect that at microscopic level nonclassical correlations between the fundamental and second-harmonic modes lead to a two-mode squeezing that arises due to the down conversion of a single-high frequency photon into a pair of correlated lower frequency photons. In connection to this, previous studies have shown the existence of a strong correlation between the fundamental and second-harmonic modes in the opposite up parametric conversion process [16]. Moreover, the inclusion of the quantum properties of the pump mode results modification in the quantum features of the cavity radiation which leads, for instance, to tripartite entanglement in a nondegenerate parametric oscillator [17].

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Basically, these are some of the motivations for studying the nonclassical properties of the light in a degenerate parametric down conversion when the pump mode is treated quantum mechanically.

It is, therefore, compelling to expect that a coherently driven degenerate parametric down conversion phenomenon can also be a source of a two-mode squeezed light characterized by a strong correlation between the fundamental and second-harmonic modes. It is believed that the squeezing can exist in each mode separately as well as in the superimposed state formed by the two modes. On the basis of the strong correlation and superposition of the two modes, the two-mode squeezed light, in general, can be used in demonstrating a variety of entangled states which currently found to have a key role in the quantum information and precession measurements in association with the preparation of Einstein-Podolsky-Rosen type entanglement [18], quantum teleportation of continuous variables [19], and testing of nonlocality [20] among others. With interest, nowadays, shifting from the production of squeezing to its application and hence attention is moving away from demonstration experiments towards producing robust, well understood sources of squeezing, it was believed that parametric amplification has enjoyed a great success as a source of continuous wave and bright squeezed light [21]. Therefore, as far as I can see, the coherently driven degenerate parametric down conversion process can also be one of the possible mechanisms for producing two-mode bright squeezed light. Hence, it goes without saying that a thorough theoretical investigation of the quantum nature of the two-mode radiation in the cavity of a degenerate parametric oscillator is required.

In line with this, the main task of this contribution is devoted to the analysis of the squeezing properties of the two-mode radiation in a cavity of a driven degenerate parametric oscillator coupled to two independent squeezed vacuum reservoirs. Owing to the fact that the differential equations following from a trilinear Hamiltonian are difficult to solve analytically, a linearization procedure in which the quantum properties of the system in time are taken to vary slightly around the steady state values is employed. A semiclassical approximation whereby the two modes are assumed to be uncorrelated at steady state is also considered. As recently discussed by Chaturvedi et al. [8], the semiclassical theory is found to work surprisingly well in the threshold region. In view of this fact, the squeezing properties of the two-mode radiation near threshold is studied by solving the linearized differential equations in the semiclassical limit following the procedure described in reference [22]. In particular, the quadrature variances for cavity mode and the squeezing spectrum for the output radiation are calculated. The obtained results are compared with the corresponding values of the fundamental mode.

2 Quantum Langevin equations

Interaction of an external coherent radiation with a nonlinear crystal responsible for the degenerate parametric oscillation placed in a resonant cavity can be described in the rotating-wave approximation and in the interaction picture by the Hamiltonian of the form [8,10]

$$\hat{H}_I = \frac{i\lambda}{2}\left[\hat{a}^{\dagger 2}\hat{b} - \hat{a}^2\hat{b}^\dagger\right] + i\varepsilon\left[\hat{b}^\dagger - \hat{b}\right], \tag{1}$$

where $\hat{a}$ and $\hat{b}$ are the time-independent annihilation operators for the fundamental (subharmonic) and pump (second-harmonic) modes, λ is the measure of the coupling of a nonlinear crystal with the cavity radiation, and ε is proportional to the amplitude of the coherent input. In this contribution, λ and ε are chosen to be real-positive constants. Now in view of the fact that $\hat{a}$ and $\hat{b}$ are mutually commuting operators, the pertinent quantum Langevin equations are found to be

$$\frac{d\hat{a}}{dt} = \lambda\hat{a}^\dagger\hat{b} - \frac{\kappa}{2}\hat{a} + \hat{F}_a(t), \tag{2}$$

$$\frac{d\hat{b}}{dt} = -\frac{\lambda}{2}\hat{a}^2 - \frac{\kappa}{2}\hat{b} + \varepsilon + \hat{F}_b(t), \tag{3}$$

where κ is a cavity damping constant taken to be the same for both modes and $\hat{F}_i(t)$, with $i = a, b$, are the Langevin noise operators satisfying, for two independent squeezed vacuum reservoirs, the correlation functions [23]

$$\langle\hat{F}_i(t)\rangle = 0, \tag{4}$$

$$\langle\hat{F}_i^\dagger(t)\hat{F}_i^\dagger(t')\rangle = \langle\hat{F}_i(t)\hat{F}_i(t')\rangle = \kappa M_i\delta(t-t'), \tag{5}$$

$$\langle\hat{F}_i^\dagger(t)\hat{F}_i(t')\rangle = \kappa N_i\delta(t-t'), \tag{6}$$

$$\langle\hat{F}_i(t)\hat{F}_i^\dagger(t')\rangle = \kappa(N_i+1)\delta(t-t'), \tag{7}$$

$$\langle\hat{F}_j^\dagger(t)\hat{F}_i(t')\rangle_{i\neq j} = \langle\hat{F}_j(t)\hat{F}_i(t')\rangle_{i\neq j} = 0, \tag{8}$$

where $N_i = \sinh^2(r_i)$ and $M_i = \sinh(r_i)\cosh(r_i)$, in which r_i is the squeeze parameter of the respective modes.

It can be noted that equations (2) and (3) are nonlinear coupled differential equations for which it is difficult to obtain the exact solutions analytically. This often leads to the application of some sort of approximation scheme. In this regard, a linearization procedure, in which

$$\hat{a}(t) = \alpha + \hat{A}(t), \tag{9}$$

$$\hat{b}(t) = \beta + \hat{B}(t), \tag{10}$$

is employed where $\hat{A}(t)$ and $\hat{B}(t)$ are very small variations about the expectation values at steady state and $\alpha = \langle\hat{a}(t)\rangle_{ss}$ and $\beta = \langle\hat{b}(t)\rangle_{ss}$. This approximation remains valid as long as the quantum fluctuations about the expectation values are much smaller than the classical mean values, namely, the quantum noise during the interaction is quite small. In addition, upon taking the statistical average of equations (2) and (3) and then applying the semiclassical approximation in which at steady state the modes are assumed to be uncorrelated, $\langle\hat{a}^\dagger(t)\hat{b}(t)\rangle_{ss} = \langle\hat{a}^\dagger(t)\rangle_{ss}\langle\hat{b}(t)\rangle_{ss}$ and $\langle\hat{a}^2(t)\rangle_{ss} = \langle\hat{a}(t)\rangle_{ss}^2$, one obtains

$$\lambda\alpha^*\beta - \frac{\kappa}{2}\alpha = 0, \tag{11}$$

$$\lambda\alpha^2 + \kappa\beta = 2\varepsilon. \tag{12}$$

Semiclassical assumption, on the other hand, is found to work for weak nonlinearity or coupling between the external radiation and nonlinear crystal, that is, the mean photon number at threshold is taken to be very large. Moreover, with the aid of equations (2), (3), (9), (10), (11), and the fact that $\hat{A}$ and $\hat{B}$ are small perturbations, it is possible to see that

$$\frac{d\hat{A}(t)}{dt} = \varepsilon_1^*\hat{B}(t) + \varepsilon_2\hat{A}^\dagger(t) - \frac{\kappa}{2}\hat{A}(t) + \hat{F}_a(t), \tag{13}$$

$$\frac{d\hat{B}(t)}{dt} = -\varepsilon_1\hat{A}(t) - \frac{\kappa}{2}\hat{B}(t) + \hat{F}_b(t), \tag{14}$$

where $\varepsilon_1^* = \lambda\alpha^*$ and $\varepsilon_2 = \lambda\beta$. On the basis of the derivation presented in Appendix A, one can readily verify that

$$\varepsilon_1 = \pm\sqrt{2\lambda\varepsilon - \kappa\varepsilon_2} \tag{15}$$

and $\varepsilon_2 = \kappa/2$ for $\varepsilon_1 \neq 0$. As can readily be seen from equation (15) the amplitude of the fundamental mode can assume two values (positive and negative), which can be interpreted as if the system is in a transient superimposed state of these amplitudes prior to detection.

Upon introducing,

$$\hat{A}_\pm = \hat{A}^\dagger \pm \hat{A}, \tag{16}$$

$$\hat{B}_\pm = \hat{B}^\dagger \pm \hat{B}, \tag{17}$$

$$\hat{E}_\pm(t) = \hat{F}_a^\dagger(t) \pm \hat{F}_a(t), \tag{18}$$

$$\hat{F}_\pm(t) = \hat{F}_b^\dagger(t) \pm \hat{F}_b(t). \tag{19}$$

One can find following the procedure described in reference [22] that the solution of equations (13) and (14) to be

$$\begin{aligned}\hat{A}_\pm(t+\tau) = &-\left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 b_\pm(\tau) + \varepsilon_2 c_\pm(\tau)\right]\\ &+ b_\pm(\tau)\hat{A}_\pm(t) + c_\pm(\tau)\hat{B}_\pm(t)\\ &+ \hat{g}_\pm(t+\tau) + \hat{f}_\pm(t+\tau),\end{aligned} \tag{20}$$

$$\begin{aligned}\hat{B}_\pm(t+\tau) = &\left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 c_\pm(\tau) - \varepsilon_2 d_\pm(\tau)\right]\\ &- c_\pm(\tau)\hat{A}_\pm(t) + d_\pm(\tau)\hat{B}_\pm(t)\\ &+ \hat{k}_\pm(t+\tau) + \hat{h}_\pm(t+\tau),\end{aligned} \tag{21}$$

where

$$b_\pm(\tau) = \frac{1}{2}\big[(1\pm p)e^{-\eta_\pm\tau} + (1\mp p)e^{-\mu_\pm\tau}\big], \tag{22}$$

$$c_\pm(\tau) = \frac{q}{2}\big[e^{-\eta_\pm\tau} - e^{-\mu_\pm\tau}\big], \tag{23}$$

$$d_\pm(\tau) = \frac{1}{2}\big[(1\mp p)e^{-\eta_\pm\tau} + (1\pm p)e^{-\mu_\pm\tau}\big], \tag{24}$$

$$\begin{aligned}\hat{f}_\pm(t+\tau) = &\frac{q}{2}\int_0^\tau \left[e^{-\eta_\pm(\tau-\tau')} - e^{-\mu_\pm(\tau-\tau')}\right]\\ &\times \hat{F}_\pm(t+\tau')d\tau',\end{aligned} \tag{25}$$

$$\begin{aligned}\hat{g}_\pm(t+\tau) = &\frac{1}{2}\int_0^\tau [(1\pm p)\, e^{-\eta_\pm(\tau-\tau')} + (1\mp p)\\ &\times e^{-\mu_\pm(\tau-\tau')}]\hat{E}_\pm(t+\tau')d\tau',\end{aligned} \tag{26}$$

$$\begin{aligned}\hat{h}_\pm(t+\tau) = &\frac{1}{2}\int_0^\tau [(1\mp p)\, e^{-\eta_\pm(\tau-\tau')} + (1\pm p)\\ &\times e^{-\mu_\pm(\tau-\tau')}]\hat{F}_\pm(t+\tau')d\tau',\end{aligned} \tag{27}$$

$$\begin{aligned}\hat{k}_\pm(t+\tau) = &\frac{q}{2}\int_0^\tau \left[e^{-\mu_\pm(\tau-\tau')} - e^{-\eta_\pm(\tau-\tau')}\right]\\ &\times \hat{E}_\pm(t+\tau')d\tau',\end{aligned} \tag{28}$$

with $\eta_\pm = \frac{\kappa\mp\varepsilon_2}{2} - \frac{1}{2}\sqrt{\varepsilon_2^2 - 4\varepsilon_1^2}$, $\mu_\pm = \frac{\kappa\mp\varepsilon_2}{2} + \frac{1}{2}\sqrt{\varepsilon_2^2 - 4\varepsilon_1^2}$, $p = \frac{\varepsilon_2}{\sqrt{\varepsilon_2^2-4\varepsilon_1^2}}$, and $q = \frac{2\varepsilon_1}{\sqrt{\varepsilon_2^2-4\varepsilon_1^2}}$. It perhaps worth mentioning that equations (20) and (21) are applied to calculate various quantities of interest in the forthcoming discussions. It is not difficult to observe that $\eta_\pm$ and $\mu_\pm$ take complex values if $\varepsilon_2 > 2\varepsilon_1$, that is, $\hat{A}_\pm(t)$ and $\hat{B}_\pm(t)$ rapidly oscillate at steady state in this case. One can also see from equation (15) that for ε_1 to be real, $2\lambda\varepsilon \geq \kappa\varepsilon_2$. Hence the case for which $2\lambda\varepsilon = \kappa\varepsilon_2$ is denoted as a threshold condition.

3 Two-mode squeezing

In this section, the squeezing of the two-mode cavity radiation that can be described by an annihilation operator,

$$\hat{c} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{b}\right), \tag{29}$$

is evaluated where $\hat{a}$ and $\hat{b}$ are the boson operators corresponding to the fundamental and second-harmonic modes. In view of the usual boson commutation relations, one can see that $\left[\hat{c},\ \hat{c}^\dagger\right] = 1$ and $\left[\hat{c},\ \hat{c}\right] = 0$. It is a well established fact that the squeezing properties of the two-mode cavity radiation can be studied using the quadrature operators associated with $\hat{c}$,

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{30}$$

and

$$\hat{c}_- = i\left(\hat{c}^\dagger - \hat{c}\right), \tag{31}$$

which can also be expressed employing equations (9), (10), (16) and (17) as

$$\hat{c}_+ = \frac{1}{\sqrt{2}}\left(2\alpha + 2\beta + \hat{A}_+ + \hat{B}_+\right), \tag{32}$$

$$\hat{c}_- = \frac{i}{\sqrt{2}}\left(\hat{A}_- + \hat{B}_-\right). \tag{33}$$

3.1 Quadrature variances

Making use of the well-known definition of the variance of an operator, the variances of the quadrature operators (32) and (33) are found to have the form

$$\begin{aligned}\Delta c_\pm^2 = \pm\frac{1}{2}\Big[&\langle \hat{A}_\pm^2(t)\rangle + \langle \hat{B}_\pm^2(t)\rangle + \langle \hat{A}_\pm(t)\hat{B}_\pm(t)\rangle \\ &+ \langle \hat{B}_\pm(t)\hat{A}_\pm(t)\rangle - \langle \hat{A}_\pm(t)\rangle^2 - \langle \hat{B}_\pm(t)\rangle^2 \\ &- 2\langle \hat{B}_\pm(t)\rangle\langle \hat{A}_\pm(t)\rangle\Big].\end{aligned} \tag{34}$$

What remains to determine is the various expectation values in equation (34). To this effect, taking both the fundamental and second-harmonic modes to be initially in a vacuum state, employing the fact that the expectation value of the noise force is zero and upon setting $t = 0$ and then replacing τ by t in equations (20) and (21), it is obtained at steady state that $\langle \hat{A}_\pm(t)\rangle_{ss} = 0$ and $\langle \hat{B}_\pm(t)\rangle_{ss} = 0$. Moreover, the other expectation values involved in equation (34) are given in Appendix B. Therefore, using equations (34), (B.5), (B.6), (B.7) and (B.8) one gets at steady state

$$\Delta c_\pm^2 = e^{\pm 2r}\left[\frac{\kappa^2(\kappa \mp 2\varepsilon_2) + \kappa\varepsilon_2^2 + 4\kappa\varepsilon_1^2 \mp 2\kappa\varepsilon_2\varepsilon_1}{(\kappa \mp \varepsilon_2)\left[\kappa(\kappa \mp 2\varepsilon_2) + 4\varepsilon_1^2\right]}\right]. \tag{35}$$

It is not difficult to see for $\lambda = 0$ that

$$\Delta c_\pm^2 = e^{\pm 2r} \tag{36}$$

and for $\lambda \neq 0$

$$\Delta c_\pm^2 = e^{\pm 2r}\left[\frac{32\lambda\varepsilon - \kappa^2(3 \pm 4) - 4\kappa\sqrt{2\lambda\varepsilon - \kappa^2/2}}{2(2 \mp 1)[8\lambda\varepsilon - \kappa^2(1 \pm 1)]}\right]. \tag{37}$$

In a similar manner one can show for fundamental mode that

$$\Delta a_\pm^2 = e^{\pm 2r}\begin{cases} 1 & \lambda = 0, \\ \dfrac{16\lambda\varepsilon - \kappa^2(2 \pm 1)}{(2 \mp 1)[8\lambda\varepsilon - \kappa^2(1 \pm 1)]} & \text{otherwise.}\end{cases} \tag{38}$$

As one can clearly see from Figure 1, the two-mode cavity radiation exhibits a considerable squeezing even when the oscillator is coupled to a vacuum reservoir. It is believed that the coherence in the external radiation before the down conversion process is responsible for the correlation between the fundamental and second-harmonic modes that leads to squeezing. This, on the other hand, indicates that even though the down conversion process splits the external coherent radiation into two fundamental modes, it is unable to destroy the existing coherence. This could also be perceived as if the phase sensitive down conversion process characteristically allocates the arising quantum noise disproportionately to the quadrature components of the superposition of the fundamental and second-harmonic modes. It turns out that the two-mode cavity radiation exhibits a significant degree of squeezing for values of $\lambda\varepsilon$ very close to $\kappa^2/4$. On the basis of equation (15) and $\varepsilon_2 = \kappa/2$ this corresponds to the value designated as a critical point. It is recalled that the mean of the annihilation operator for the fundamental mode at steady state is zero at the critical point. It is also observed that, for a fixed value of the coupling constant ($\lambda = 0.5$), the squeezing increases with the amplitude of the external radiation for smaller values of $\lambda\varepsilon$. Moreover, it is found that the degree of achievable squeezing for the two-mode cavity radiation can reach up to 42% for a vacuum reservoir, $\kappa = 0.5$, $\lambda = 0.5$, and $\varepsilon = 0.2$. It is also directly evident from Figure 1 and equation (37) that the squeezing is significantly enhanced by the biased noise fluctuations of the squeezed vacuum reservoir modes, although the two reservoirs are assumed to be independent. In this respect, further manipulation reveals that a squeezing of 87% occurs for $r = 0.75$, $\kappa = 0.5$, $\lambda = 0.5$, and $\varepsilon = 0.2$.

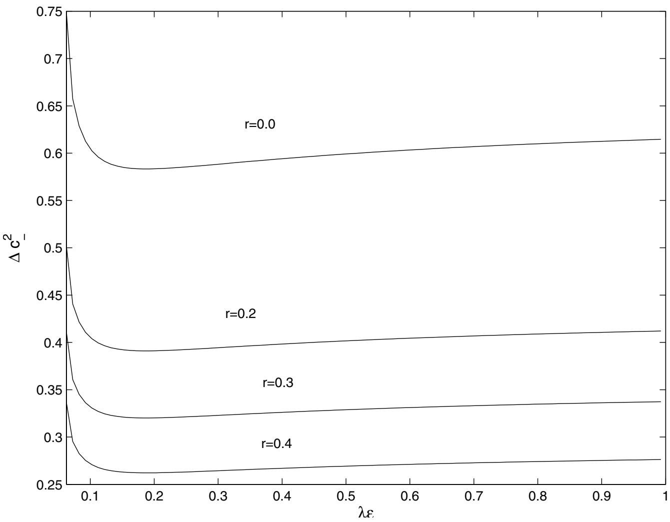

Fig. 1. Plots of the minus quadrature variance for the two-mode cavity radiation (Eq. (37)) at steady state for $\kappa = 0.5$, $\lambda = 0.5$ and different values of r.

It is indicated in Figure 2 that the fundamental mode of the driven degenerate parametric oscillator exhibits squeezing in a different manner from the two-mode cavity radiation, of course, with differing degree of squeezing. It is found that 50% degree of squeezing occurs for $r = 0$, $\lambda = 0.5$, $\kappa = 0.5$, and $\varepsilon = 0.0625$. The same result has been reported, at critical point, by various authors using different approaches [1–4]. As in the case of the two-mode radiation, the presence of the squeezed vacuum reservoir outside the cavity significantly improves the squeezing of

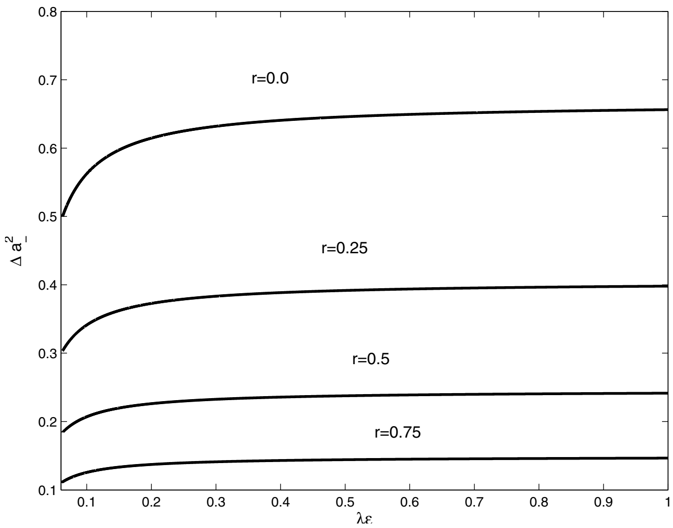

Fig. 2. Plots of the minus quadrature variance for the fundamental mode (Eq. (38)) at steady state for $\kappa = 0.5$, $\lambda = 0.5$, and different values of r.

the fundamental mode. It is found that 89% of squeezing is achievable for $r = 0.75$, $\lambda = 0.5$, $\kappa = 0.5$, and $\varepsilon = 0.0625$. Comparison of Figures 1 and 2 reveals that the two-mode cavity radiation exhibits slightly smaller maximum squeezing than the fundamental mode. However, one can readily see from equations (37) and (38) that, for $\lambda = 0$, the squeezing in both cases reduces to that of ordinary squeezed vacuum. That is, for $r = 0$, the squeezing is not observed in the cavity, since there is no down conversion of photons for $\lambda = 0$. One can, hence, deduce that the cause for the squeezing in the fundamental mode and two-mode cavity radiation is the correlation initiated in the process of parametric down conversion.

3.2 Squeezing spectrum

The squeezing spectrum of the output of a two-mode radiation can be expressed as

$$S^{out}_{c\pm}(\omega) = 2\mathrm{Re}\int_0^\infty e^{i\omega\tau}\left\langle \hat{c}^{out}_\pm(t+\tau),\ \hat{c}^{out}_\pm(t)\right\rangle_{ss} d\tau, \quad (39)$$

where

$$\langle \hat{c}^{out}_\pm(t+\tau),\ \hat{c}^{out}_\pm(t)\rangle = \langle \hat{c}^{out}_\pm(t+\tau)\hat{c}^{out}_\pm(t)\rangle - \langle \hat{c}^{out}_\pm(t+\tau)\rangle\langle \hat{c}^{out}_\pm(t)\rangle. \quad (40)$$

Now making use of the input-output relations introduced by Gardiner and Collett [24],

$$\hat{a}^{out}(t) = \sqrt{\kappa}\hat{a}(t) - \frac{1}{\sqrt{\kappa}}\hat{F}_a(t), \quad (41)$$

$$\hat{b}^{out}(t) = \sqrt{\kappa}\hat{b}(t) - \frac{1}{\sqrt{\kappa}}\hat{F}_b(t), \quad (42)$$

one finds

$$\hat{c}^{out}_+(t) = \frac{1}{\sqrt{2}}\Big[2\alpha^{out} + 2\beta^{out} + \sqrt{\kappa}(\hat{A}_+ + \hat{B}_+) - \frac{1}{\sqrt{\kappa}}(\hat{E}_+ + \hat{F}_+)\Big], \quad (43)$$

$$\hat{c}^{out}_-(t) = \frac{i}{\sqrt{2}}\left[\sqrt{\kappa}(\hat{A}_- + \hat{B}_-) - \frac{1}{\sqrt{\kappa}}(\hat{E}_- + \hat{F}_-)\right]. \quad (44)$$

In relation to the fact that the noise force at $t+\tau$ is not correlated with the system variables at t and taking equations (40), (43) and (44) into consideration, one can readily see that

$$\langle \hat{c}^{out}_\pm(t+\tau),\ \hat{c}^{out}_\pm(t)\rangle = \pm\frac{1}{2}\big\langle \kappa(A_\pm(t+\tau) + B_\pm(t+\tau))(A_\pm(t) + B_\pm(t)) + \frac{1}{\kappa}(E_\pm(t+\tau) + F_\pm(t+\tau))(E_\pm(t) + F_\pm(t)) - (A_\pm(t+\tau) + B_\pm(t+\tau))(E_\pm(t) + F_\pm(t)\big\rangle. \quad (45)$$

Next in view of equations (20), (21), (22), (23) and (24), one reaches, for $r_a = r_b$, at

$$\langle \hat{A}_\pm(t+\tau)\hat{A}_\pm(t)\rangle_{ss} + \langle \hat{B}_\pm(t+\tau)\hat{B}_\pm(t)\rangle_{ss} + \langle \hat{A}_\pm(t+\tau)\hat{B}_\pm(t)\rangle_{ss} + \langle \hat{B}_\pm(t+\tau)\hat{A}_\pm(t)\rangle_{ss} = \frac{1}{2}\big[(X_\pm + Y_\pm)e^{-\eta_\pm\tau} + (X_\pm - Y_\pm)e^{\mu_\pm\tau}\big], \quad (46)$$

where

$$X_\pm = \frac{1}{2}\left[\langle \hat{A}^2_\pm(t)\rangle_{ss} + \langle \hat{B}^2_\pm(t)\rangle_{ss} + 2\langle \hat{A}_\pm(t)\hat{B}_\pm(t)\rangle_{ss}\right], \quad (47)$$

$$Y_\pm = \frac{1}{2}\left[-(q \mp p)\langle \hat{A}^2_\pm(t)\rangle_{ss} + (q \mp p\langle \hat{B}^2_\pm(t)\rangle_{ss}\right]. \quad (48)$$

On the other hand, on account of the correlation functions (5), (6), (7) and (8), one gets

$$\langle \hat{E}_\pm(t+\tau)\hat{E}_\pm(t)\rangle_{ss} + \langle \hat{E}_\pm(t+\tau)\hat{F}_\pm(t)\rangle_{ss} + \langle \hat{F}_\pm(t+\tau)\hat{E}_\pm(t)\rangle_{ss} + \langle \hat{F}_\pm(t+\tau)\hat{F}_\pm(t)\rangle_{ss} = 2\kappa(2M \pm 2N \pm 1)\delta(\tau). \quad (49)$$

Furthermore, application of equations (20), (21), (22), (23) and (24) leads to

$$\langle \hat{A}_\pm(t+\tau)\hat{E}_\pm(t)\rangle = \frac{\kappa}{2}(2M \pm 2N \pm 1) \times \left[(1 \pm p)e^{-\eta_\pm\tau} + (1 \mp p)e^{-\mu_\pm\tau}\right], \quad (50)$$

$$\langle \hat{A}_{\pm}(t+\tau)\hat{F}_{\pm}(t)\rangle = \frac{\kappa}{2}(2M \pm 2N \pm 1)q \times \left[e^{-\eta_{\pm}\tau} - e^{-\mu_{\pm}\tau}\right], \quad (51)$$

$$\langle \hat{B}_{\pm}(t+\tau)\hat{F}_{\pm}(t)\rangle = \frac{\kappa}{2}(2M \pm 2N \pm 1) \times \left[(1 \mp p)e^{-\eta_{\pm}\tau} + (1 \pm p)e^{-\mu_{\pm}\tau}\right], \quad (52)$$

$$\langle \hat{B}_{\pm}(t+\tau)\hat{E}_{\pm}(t)\rangle = -\frac{\kappa}{2}(2M \pm 2N \pm 1)q \times \left[e^{-\eta_{\pm}\tau} - e^{-\mu_{\pm}\tau}\right]. \quad (53)$$

Hence using equations (45), (46), (49), (50), (51), (52) and (53), one obtains

$$\langle \hat{c}^{out}_{\pm}(t+\tau),\ \hat{c}^{out}_{\pm}(t)\rangle = \big(2M \pm 2N \pm 1\big)\delta(\tau) + \kappa\big(X_{\pm} + Y_{\pm} - (2M \pm 2N \mp 1)\big)e^{-\eta_{\pm}\tau} + \kappa\big(X_{\pm} - Y_{\pm} - (2M \pm 2N \pm 1)\big)e^{-\mu_{\pm}\tau}, \quad (54)$$

in which the squeezing spectrum of the two-mode radiation takes the form

$$S^{out}_{c\pm}(\omega) = 1 + 2N \pm 2M \pm \kappa\Bigg[\frac{\eta_{\pm}}{\eta_{\pm}^2 + \omega^2}\big(X_{\pm} + Y_{\pm} - (2M \pm 2N \pm 1)\big) + \frac{\mu_{\pm}}{\mu_{\pm}^2 + \omega^2}\big(X_{\pm} + Y_{\pm} - (2M_b \pm 2N_b \pm 1)\big)\Bigg]. \quad (55)$$

Following a similar approach, the squeezing spectrum of the fundamental mode turns out to be

$$S^{out}_{a\pm}(\omega) = 1 + 2N_a \pm 2M_a \pm \kappa\Bigg[\frac{\eta_{\pm}}{\eta_{\pm}^2 + \omega^2}\big(\langle \hat{A}^2_{\pm}(t)\rangle_{ss} + R_{\pm} - (2M_a \pm 2N_a \pm 1)(1 \pm p)\big) + \frac{\mu_{\pm}}{\mu_{\pm}^2 + \omega^2} \times \big(\langle \hat{A}^2_{\pm}(t)\rangle_{ss} - R_{\pm} - (2M_a \pm 2N_a \pm 1)(1 \mp p)\big)\Bigg], \quad (56)$$

where $R_{\pm} = \pm p\langle \hat{A}^2_{\pm}(t)\rangle_{ss} + q\langle \hat{B}_{\pm}(t)\hat{A}_{\pm}(t)\rangle_{ss}$. It is not difficult to see from equations (55) and (56) that the output squeezing would be maximum for $\omega = 0$, that is why I choose $\omega = 0$ in the following plots.

It is not difficult to see from Figure 3 that the output of the two-mode radiation exhibits substantial degree of squeezing for certain values of the amplitude of the driving radiation given that the coupling constant is independent of the amplitude. It is found that a better squeezing is achievable for $\kappa = 0.5$, $\lambda = 0.5$, and $\varepsilon = 0.095$. Like the case inside the cavity, the maximum squeezing for the output radiation occurs just above the critical point, which agrees with the prediction of Drummond et al. [10] for second-harmonic mode. It can also be seen from Figure 4 that there is quenching of the noise in the minus quadrature component for the output radiation of the fundamental mode irrespective of the values of the squeeze parameter. This quenching of the noise for the fundamental mode has been known since the work of Milburn and Walls [1] and similar result has also recently reported [6]. Unfortunately, this prediction is clearly unrealistic, since by virtue of the Heisenberg uncertainty principle this necessarily requires an infinite noise fluctuations in the conjugate quadrature component. Moreover, it is observed that the squeezing of the output radiation for both fundamental mode and two-mode radiation is significantly improved by the squeezed vacuum reservoir. It is a well-known fact that in determining the squeezing spectrum, the correlation between the quadrature operators at different times

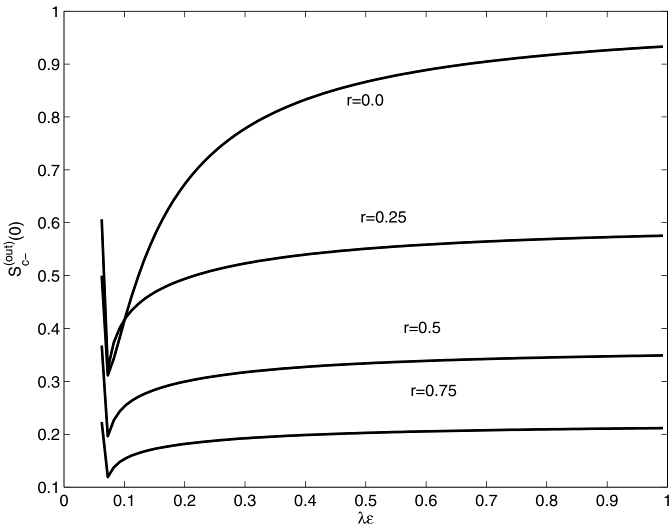

Fig. 3. Plots of the squeezing spectrum (Eq. (55)) for the two-mode cavity radiation for $\kappa = 0.5$, $\omega = 0$, $\lambda = 0.5$ and different values of r.

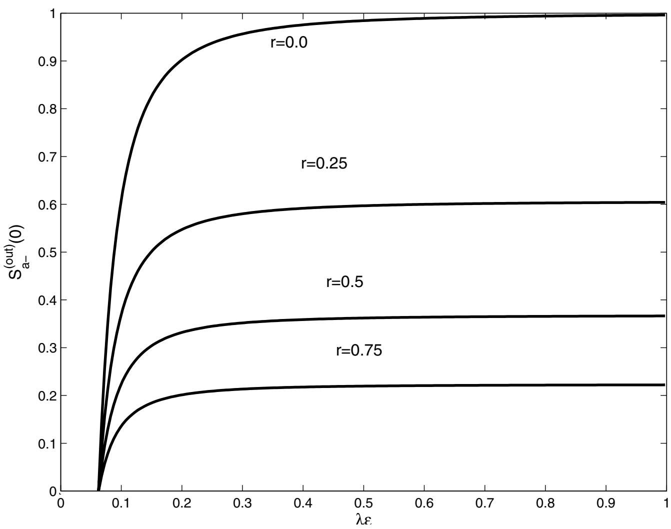

Fig. 4. Plots of the squeezing spectrum (Eq. (56)) for the fundamental mode for $\kappa = 0.5$, $\omega = 0$, $\lambda = 0.5$ and different values of r.

are evaluated in the frequency domain. It is, therefore, realized that the quenching of the overall noise in one of the quadrature components of the fundamental mode and the increment of the squeezing for two-mode radiation for small amplitude of the driving radiation can be related to this approach.

4 Conclusion

The squeezing properties of the two-mode radiation and fundamental mode in a driven degenerate parametric down conversion process when the cavity is coupled to two independent squeezed vacuum reservoirs inside and outside the cavity, are thoroughly analyzed. The resulting nonlinear differential equations are solved applying a linearization procedure in the semiclassical approximation. It turns out that both the two-mode radiation and the fundamental mode exhibit significant squeezing inside and outside the cavity. Though the squeezing of the fundamental mode alone has attracted a great deal of interest in the previous studies, it is found that the degree of squeezing of the two-mode radiation is also as nearly considerable as the fundamental mode. In addition to this, since the squeezing in the two-mode radiation corresponds to the strong correlation between the fundamental and second-harmonic modes, it is found recently that an entanglement and Einstein-Podolsky-Rosen type correlation between these radiations can exist [7]. It is not difficult to realize that the squeezing can be maximized by properly selecting the values of the damping and coupling constants along with the amplitude of the driving radiation. However, the maximum achievable squeezing in the cavity radiation in the absence of the squeezed vacuum reservoir is limited to 42% and 50% for two-mode radiation and fundamental mode, respectively due to the leakage through the mirror and amplification of the quantum noise. Moreover, it can easily be inferred from equations (37) and (38) as well as Figures 1 and 2 that the squeezing of the fundamental mode and the two-mode radiation in the cavity is exponentially enhanced by the squeezed input.

The degree of squeezing is found to decrease for fundamental mode and increase for two-mode radiation with the amplitude of the driving radiation, if the coupling constant is taken to be independent of the amplitude of the external radiation near critical point. The maximum cavity squeezing for the fundamental mode occurs when $\varepsilon_1 = 0$, which corresponds to the situation in which the expectation value of the annihilation operator for the fundamental mode is zero at steady state. However, the maximum squeezing for the two-mode radiation occurs slightly above this critical point. Hence the maximum squeezing for the output two-mode radiation occurs for the amplitude of the driving radiation slightly greater than the critical value. It can be deduced from what is discussed so far that the correlation between the states of the down converted photons is nearly the same as the correlation between the fundamental and second-harmonic modes. It can also observed from the results obtained that though the down conversion process splits the external coherent radiation, it is unable to destroy the coherence that is responsible for the correlation which leads to squeezing.

Appendix A: Proof of $\alpha = \alpha^*$ and $\beta = \beta^*$

In this appendix, I show that $\alpha = \alpha^*$ and $\beta = \beta^*$. To this end, upon multiplying equations (11) and (12) by λ, results

$$\varepsilon_1^*\varepsilon_2 - \frac{\kappa}{2}\varepsilon_1 = 0, \tag{A.1}$$

$$\varepsilon_1^2 + \kappa\varepsilon_2 = 2\lambda\varepsilon, \tag{A.2}$$

from which readily follows

$$\varepsilon_1 = \frac{2\varepsilon_1^*\varepsilon_2}{\kappa}. \tag{A.3}$$

In addition, multiplying equation (A.2) by ε_1^* and then inserting the complex conjugate of equation (A.3) in the resulting expression, it is possible to get

$$\varepsilon_1^*\varepsilon_1^2 + 2\varepsilon_1\varepsilon_2^*\varepsilon_2 - 2\lambda\varepsilon\varepsilon_1^* = 0. \tag{A.4}$$

Now subtracting equation (A.4) from its complex conjugate leads to

$$(\varepsilon_1^*\varepsilon_1 + 2\varepsilon_2^*\varepsilon_2 + 2\lambda\varepsilon)(\varepsilon_1 - \varepsilon_1^*) = 0, \tag{A.5}$$

which holds true if $\varepsilon_1 = \varepsilon_1^*$. Therefore, it is not difficult to see from equation (A.3) that $\varepsilon_2 = \varepsilon_2^*$. Hence on the basis of these facts, it can be noted that $\alpha = \alpha^*$ and $\beta = \beta^*$, since λ is taken to be positive constant.

Appendix B: Various correlations

In here various correlations are determined. To this end, with the aid of equations (25) and (26) along with the fact that the noise force at latter time is not correlated with the system variables at the earlier times, one gets

$$\begin{aligned}\langle \hat{A}_\pm^2(t)\rangle = -\left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 b_\pm(t) + \varepsilon_2 c_\pm(t)\right]\langle \hat{A}_\pm(t)\rangle \\ \pm\left[b_\pm^2(t) + c_\pm^2(t)\right] + \langle \hat{g}_\pm^2(t)\rangle + \langle \hat{f}_\pm^2(t)\rangle \\ + 2\langle \hat{g}_\pm(t)\hat{f}_\pm(t)\rangle.\end{aligned} \tag{B.1}$$

Besides, on account of equations (5), (6), (7), (8), (18), (19), (25) and (26), one finds

$$\begin{aligned}\langle \hat{g}_\pm^2(t)\rangle = \frac{\kappa(2M_a \pm 2N_a \pm 1)}{4}\Bigg[\frac{(1\pm p)^2}{2\eta_\pm}\big(1 - e^{-2\eta_\mp t}\big) \\ +\frac{(1\mp p)^2}{2\mu_\mp}\big(1 - e^{-2\mu_\pm t}\big) \\ +\frac{2(1-p^2)}{\eta_\pm + \mu_\pm}\big(1 - e^{-(\eta_\pm+\mu_\pm)t}\big)\Bigg],\end{aligned} \tag{B.2}$$

$$\langle \hat{f}^2_\pm(t)\rangle = \frac{\kappa(2M_b \pm 2N_b \pm 1)}{4}\left[\frac{q^2}{2\eta_\pm}\left(1 - e^{-2\eta_\mp t}\right) + \frac{q^2}{2\mu_\pm}\left(1 - e^{-2\mu_\pm t}\right) - \frac{2q^2}{\eta_\pm + \mu_\pm}\left(1 - e^{-(\eta_\pm+\mu_\pm)t}\right)\right], \quad \text{(B.3)}$$

$$\langle \hat{g}_\pm(t)\hat{f}_\pm(t)\rangle = 0. \quad \text{(B.4)}$$

Upon taking the squeeze parameter of the two reservoir modes to be the same ($r_a = r_b$), it is not difficult to arrive at

$$\langle \hat{A}^2_\pm(t)\rangle = -\left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 b_\pm(t) + \varepsilon_2 c_\pm(t)\right]\langle \hat{A}_\pm(t)\rangle \pm \left[b^2_\pm(t) + c^2_\pm(t)\right] + \frac{\kappa(2M \pm 2N \pm 1)}{4} \times \left[\frac{1 + p^2 + q^2 \pm 2p}{2\eta_\pm}\left(1 - e^{-2\eta_\pm t}\right) + \frac{1 + p^2 + q^2 \mp 2p}{2\mu_\pm}\left(1 - e^{-2\mu_\pm t}\right) + \frac{2(1 - p^2 - q^2)}{\eta_\pm + \mu_\pm}\left(1 - e^{-(\eta_\pm+\mu_\pm)t}\right)\right]. \quad \text{(B.5)}$$

Following a similar approach, it is possible to verify that

$$\langle \hat{B}^2_\pm(t)\rangle = \left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 c_\pm(t) - \varepsilon_2 d_\pm(t)\right]\langle \hat{B}_\pm(t)\rangle \pm \left[b^2_\pm(t) + c^2_\pm(t)\right] + \frac{\kappa(2M \pm 2N \pm 1)}{4} \times \left[\frac{1 + p^2 + q^2 \mp 2p}{2\eta_\pm}\left(1 - e^{-2\eta_\pm t}\right) + \frac{1 + p^2 + q^2 \pm 2p}{2\mu_\pm}\left(1 - e^{-2\mu_\pm t}\right)\right] + \frac{2(1 - p^2 - q^2)}{\eta_\pm + \mu_\pm}\left(1 - e^{-(\eta_\pm+\mu_\pm)t}\right)\Bigg], \quad \text{(B.6)}$$

$$\langle \hat{A}_\pm(t)\hat{B}_\pm(t)\rangle = -\left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 b_\pm(t) + \varepsilon_2 c_\pm(t)\right]\langle \hat{B}_\pm(t)\rangle + c_\pm(t)\left[d_\pm(t) - b_\pm(t)\right] \mp \frac{qp}{4}(2M \pm 2N \pm 1) \times \left[\frac{1}{\eta_\pm}\left(1 - e^{-2\eta_\pm t}\right) + \frac{1}{\mu_\pm}\left(1 - e^{-2\mu_\pm t}\right) - \frac{2}{(\eta_\pm + \mu_\pm)}\left(1 - e^{-(\eta_\pm+\mu_\pm)t}\right)\right], \quad \text{(B.7)}$$

$$\langle \hat{B}_\pm(t)\hat{A}_\pm(t)\rangle = \left(\frac{1\pm 1}{\lambda}\right)\left[\varepsilon_1 c_\pm(t) - \varepsilon_2 d_\pm(t)\right]\langle \hat{A}_\pm(t)\rangle + c_\pm(t)\left[d_\pm(t) - b_\pm(t)\right] \mp \frac{qp}{4}(2M \pm 2N \pm 1) \times \left[\frac{1}{\eta_\pm}\left(1 - e^{-2\eta_\pm t}\right) + \frac{1}{\mu_\pm}\left(1 - e^{-2\mu_\pm t}\right) - \frac{2}{(\eta_\pm + \mu_\pm)}\left(1 - e^{-(\eta_\pm+\mu_\pm)t}\right)\right]. \quad \text{(B.8)}$$

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